Smooth branes and junction conditions in Einstein Gauss-Bonnet gravity

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Abstract

Using "smooth brane" solutions of the field equations, we give an alternative derivation of the junction conditions for a "brane" in a five dimensional "bulk", when gravity is governed by the Einstein Lanczos (Gauss-Bonnet) equations.

I. Introduction

Higher dimensional gravity theories based on the Lanczos Lagrangian or its generalization by Lovelock (also called Gauss-Bonnet or Euler Lagrangians), which are non-linear in the curvature but such that the field equations remain second order in the metric coefficients, have been known for a long time, see [1] for early references and a recent review.

They have recently attracted renewed interest motivated by the invention of "brane scenarios" in which the observable universe is described as a four dimensional singular surface, or "brane", of a five dimensional space-time, or "bulk" obeying Einstein's equations for gravity (see [2] for basic references and a recent review).

The extension of these brane models to gravity theories based on the Einstein Gauss-Bonnet Lagrangian (see [3-4]) has however been plagued by the problem of generalizing the Israel junction conditions [5] which describe gravity on the brane. The reason for the difficulty is that the field equations are only quasi-linear in the second order derivatives of the metric coefficients (see e.g. [1]). As a result conflicting linearized equations for brane gravity [3-4] and conflicting cosmological models [6-8] can be found in the recent literature. A physical explanation for these differences has been given in [10] using thick brane models.

In this contribution, using an adequate definition of the brane stress energy tensor, we confirm the results obtained in [9] (which agree with [4] and [6] and generalize the "total bending" junction conditions of [10]). To do so we use an approach, directly based on the field equations rather than on considerations on the proper boundary terms to be added to the action. More precisely we consider smooth "brane" solutions of the field equations for gravity coupled to a confining scalar field and show that they tend, in the limit of infinite thinness, to a solution for a thin brane endowed with matter whose stress energy tensor is the one given by the general junction conditions obtained in [9].

II. The thin brane problem in Einstein Gauss-Bonnet theory : a summary

To construct a "Z_2-symmetric braneworld" one considers a 5-dimensional manifold V_+ with a timelike edge ; one then makes a copy V_- of V_+ and superposes the copy and the original manifold onto each other along the edge (this is the so-called Z_2 symmetry) ; one thus obtains a spacetime, or braneworld, composed of a "bulk" V_5, and a singular surface, or "brane" Σ_4, whose extrinsic curvature is discontinuous : the extrinsic curvature of Σ_4 embedded in V_- is the opposite of its extrinsic curvature as embedded in V_+.

Suppose now that gravity in the bulk V_5 is described by the vacuum Einstein Lanczos (Gauss-Bonnet) equations, that is

$$\sigma^A_{[2]B} \equiv \Lambda\delta^A_B + \sigma^A_B + \alpha\,\sigma^A_{(2)B} = 0 \qquad (2.1)$$

Λ being the bulk "cosmological constant", $\alpha(>0)$ some $(length)^2$ parameter and the Einstein and Lanczos tensors being defined as

$$\sigma^A_B \equiv r^A_B - \frac{1}{2}\delta^A_B\, s\,,$$

$$\sigma^A_{(2)B} \equiv 2\left[R^{ALMN}R_{BLMN} - 2r^{LM}R^A{}_{LBM} - 2r^L_A r_{LB} + sr^A_B\right] - \frac{1}{2}\delta^A_B L_{(2)}\,, \tag{2.2}$$

$$L_{(2)} \equiv s^2 - 4r^{LM}r_{LM} + R^{LMNP}R_{LMNP}$$

where $R^A{}_{BCD} \equiv \partial_C\Gamma^A_{BD} - ...$ are the components of the Riemann tensor, Γ^A_{BD} being the Christoffel symbols, all indiced being moved with the metric g_{AB} and its inverse g^{AB} ; $r_{BD} \equiv R^A{}_{BAD}$ are the Ricci tensor components, $s \equiv g^{BD}r_{BD}$ is the scalar curvature.

Suppose also, for definitiveness, that the bulk is locally anti-de Sitter spacetime. Then, because of maximal symmetry,

$$R_{ABCD} = -\frac{1}{\mathcal{L}^2}(g_{AC}g_{BD} - g_{BC}g_{AD}) \tag{2.3}$$

with the characteristic length scale $\mathcal{L}$ given by

$$\frac{1}{\mathcal{L}^2} = \frac{1}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda}{3}}\right) \tag{2.4}$$

in order to satisfy (2.1).

Finally suppose, for the sake of the argument, that Σ_4 is flat. A convenient coordinate system to describe the almost everywhere anti-de Sitter braneworld is, in that case

$$ds^2|_5 = dw^2 + e^{-2|w|/\mathcal{L}}\eta_{\mu\nu}dx^\mu dx^\nu \tag{2.5}$$

with $\eta_{\mu\nu} = (-,+,+,+)$ the Minkowski metric, where $w > 0$ spans V_+ and $w < 0$ spans V_- and where the brane is located at $w = 0$ (and $\mathcal{L} > 0$).

The extrinsic curvature of Σ_4 in $V_5 \sqcup \Sigma_4$ is discontinuous :

$$K_{\mu\nu} = \frac{1}{\mathcal{L}}\eta_{\mu\nu}\,\mathcal{S}(w) \tag{2.6}$$

where the sign distribution $\mathcal{S}(w)$ is +1 if $w > 0$, and -1 if $w < 0$. Some components of the braneworld Riemann tensor therefore exhibit a delta-type singularity (since $\mathcal{S}'(w) = 2\delta(w)$) and one expects that the braneworld $V_5 \sqcup \Sigma_4$ satisfies the Einstein Lanczos (Gauss-Bonnet) equations everywhere—that is, Σ_4 included— but in the presence of "matter" localised on Σ_4, i.e. that one has, in $V_5 \sqcup \Sigma_4$:

$$\sigma^A_{[2]B} = T^A_B\,\mathcal{D}(w) \tag{2.7}$$

where $\mathcal{D}$ is a distribution localized on Σ_4, i.e. proportional to some linear combination of the Dirac delta distribution and its derivatives and where T^A_B is interpreted as the stress-energy tensor of "tension plus matter" in the brane. If $\mathcal{D}$ is the Dirac distribution (and we shall see that this is indeed the case) then (2.7) relates the "total bending" of the brane to its stress-energy tensor, as [10]

$$\int_{-\infty}^{+\infty}\sigma^A_{[2]B}\,dw = T^A_B\,.$$

The question now is to express this "stress-energy" tensor in terms of the discontinuity of the extrinsic curvature.

In the simple example of a flat brane in AdS$_5$ it is a straightforward exercise to find that $\sigma^\mu_{[2]\nu}$ (and only $\sigma^\mu_{[2]\nu}$) possesses a part which is confined on Σ_4 :

$$\sigma^\mu_{[2]\nu} = -6\,\delta^\mu_\nu\,\frac{\delta(w)}{\mathcal{L}}\left[1 - 4\frac{\mathcal{S}^2(w)}{\mathcal{L}^2}\right]\,. \tag{2.8}$$

Hence, in pure Einstein theory ($\alpha = 0$, $\frac{6}{\mathcal{L}^2} = -\Lambda$) one recovers the well-known result [2]

$$T^w_w = T^w_\mu = 0 \quad , \quad T^\mu_\nu = -\frac{6}{\mathcal{L}}\delta^\mu_\nu \tag{2.9}$$

which is nothing but the Israel junction conditions [5] applied to the problem at hand.

When $\alpha \neq 0$, the product of the Dirac and the sign distribution squared is not straightforwardly defined. Indeed, with $\mathcal{S}^2 = 1$ in a distributional sense and $\frac{1}{2}\mathcal{S}' = \delta$, the question is to know whether $\delta\mathcal{S}^2 = \frac{1}{2}\mathcal{S}'\mathcal{S}^2$ is equal to δ or, using the Leibniz rule, to $\frac{1}{3}\delta$, and various proposals have been put forward to give a meaning to (2.8), see [3-4] [6-8] and [1] for a review. They all boil down to obtaining

$$T^w_w = T^w_\mu = 0 \quad , \quad T^\mu_\nu = -\frac{6}{\mathcal{L}}\delta^\mu_\nu\left(1 - C\frac{4\alpha}{\mathcal{L}^2}\right) \tag{2.10}$$

with either $C = 1$ (see, e.g, [7]), $C = \frac{1}{3}$ (see, e.g., [6]) or C a constant which, it is argued, may depend on the microphysics of the brane, see [8].

When the brane is flat, the difference between the various proposals is immaterial as it amounts to different normalisations of the brane tension. But it matters when one treats cosmological models for example. Indeed (see the review in [1] for details) one gets for the tension plus matter energy density $\rho \equiv -T^0_0$ the following, very different, results, depending on whether one has chosen $C = 1$ or $C = \frac{1}{3}$ in (2.10) :

$$\rho = 6\left(1 - \frac{4\alpha}{\mathcal{L}^2}\right)\sqrt{h^2 + \frac{\kappa}{a^2} + \frac{1}{\mathcal{L}^2}} \qquad \text{if} \qquad C = 1 \tag{2.11}$$

or

$$\rho = 6\left[1 - \frac{4\alpha}{\mathcal{L}^2} + \frac{8\alpha}{3}\left(h^2 + \frac{\kappa}{a^2} + \frac{1}{\mathcal{L}^2}\right)\right]\sqrt{h^2 + \frac{\kappa}{a^2} + \frac{1}{\mathcal{L}^2}} \qquad \text{if} \qquad C = \frac{1}{3} \tag{2.12}$$

where $a(t)$ is the scale factor of the Friedmann-Lemaître brane, where $\kappa = +1, 0, -1$ characterizes its spatial curvature and $h \equiv \frac{\dot{a}}{a}$ is its Hubble parameter.

Now, from considerations on the proper boundary terms to be added to the action yielding the field equations (2.1), Davis and Gravanis-Willison [9] gave a general expression of the stress-energy tensor of matter plus tension on the brane, in terms of its extrinsic curvature in V_+ and its intrinsic Riemann tensor.

More precisely these authors associate to a braneworld the following action

$$S = \int_{V_5} d^5x\sqrt{-g}\,(-2\Lambda + s + \alpha L_{(2)}) + 2\int_{\Sigma_4} d^4x\sqrt{-\bar{g}}\,\mathcal{L}_m - 2\int_{\Sigma_4} d^4x\sqrt{-\bar{g}}\,Q\,. \tag{2.13}$$

g is the determinant of the bulk metric coefficients g_{AB}, $\bar{g}$ that of the induced brane metric coefficients $\bar{g}_{\mu\nu}$. In the second term, $\mathcal{L}_m(\bar{g}_{\mu\nu})$ is the brane "tension plus matter" Lagrangian. The third, boundary, term is [9]

$$Q \equiv 2K + 4\alpha(J - 2\bar{\sigma}^{\mu}_{\nu}\, K^{\nu}_{\mu}) \tag{2.14}$$

with J the trace of

$$J^{\mu}_{\nu} \equiv -\frac{2}{3}K^{\mu}_{\rho}K^{\rho}_{\sigma}K^{\sigma}_{\nu} + \frac{2}{3}K\, K^{\mu}_{\rho}K^{\rho}_{\nu} + \frac{1}{3}K^{\mu}_{\nu}(K.K - K^2)\,. \tag{2.15}$$

$\bar{\sigma}^{\mu}_{\nu} \equiv \bar{r}^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\bar{s}$ is the intrinsic Einstein tensor of the brane Σ_4 and K^{μ}_{ν} its extrinsic curvature in V_+, all indices μ being moved with $\bar{g}_{\mu\nu}$ and its inverse $\bar{g}^{\mu\nu}$.

Thanks to this boundary term the variation of S with respect to the metric coefficients is given in terms of their variations only as [9] :

$$\delta S = \int_{V_5} d^5x\sqrt{-g}\,\sigma^{[2]}_{AB}\,\delta g^{AB} + \int_{\Sigma_4} d^4x\sqrt{-\bar{g}}\,(2B_{\mu\nu} - T_{\mu\nu})\,\delta\bar{g}^{\mu\nu}\,. \tag{2.16}$$

The "braneworld equations of motion" are therefore $\delta S = 0$, with the metric fixed at the boundaries at infinity only. They are, first, the Einstein Gauss-Bonnet "bulk" equations (2.1) and, second, the brane equations, which generalize the Israel junction conditions to Einstein Gauss-Bonnet gravity :

$$B^{\mu}_{\nu} \equiv K^{\mu}_{\nu} - K\delta^{\mu}_{\nu} + 4\alpha\left(\frac{3}{2}J^{\mu}_{\nu} - \frac{1}{2}J\delta^{\mu}_{\nu} - \bar{P}^{\mu}{}_{\rho\nu\sigma}K^{\rho\sigma}\right) = \frac{1}{2}T^{\mu}_{\nu} \tag{2.17}$$

where

$$\bar{P}_{\mu\rho\nu\sigma} \equiv \bar{R}_{\mu\rho\nu\sigma} + (\bar{r}_{\mu\sigma}\bar{g}_{\rho\nu} - \bar{r}_{\rho\sigma}\bar{g}_{\mu\nu} + \bar{r}_{\rho\nu}\bar{g}_{\mu\sigma} - \bar{r}_{\mu\nu}\bar{g}_{\rho\sigma}) - \frac{1}{2}\bar{s}\,(\bar{g}_{\mu\sigma}\bar{g}_{\rho\nu} - \bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}) \tag{2.18}$$

and where $T_{\mu\nu}$ is defined by $\delta(\sqrt{-\bar{g}}\mathcal{L}_m) \equiv -\frac{1}{2}\sqrt{-\bar{g}}\,T_{\mu\nu}\,\delta\bar{g}^{\mu\nu}$ and is interpreted as the stress-energy tensor of "tension plus matter" on the brane.

When applied to a Friedmann-Lemaître (or flat) brane the brane equations (2.17) reduce to the "$C = 1/3$" result (2.12).

The purpose of this contribution is to confirm this "$C = 1/3$" result using the field equations and the definition of the stress-energy tensor as equal to the "total brane bending" (eq. 2.7).

III. A smooth flat brane toy model

Consider a five dimensional spacetime obeying the Einstein Gauss-Bonnet equations with matter, that is such that

$$\sigma^{A}_{[2]B} = \mathcal{T}^{A}_{B} \tag{3.1}$$

where the Einstein Lanczos tensor is defined in (2.1-2) and where we consider matter to be a scalar field $\phi(x^A)$ with potential $V(\phi)$ and stress energy tensor

$$\mathcal{T}_{AB} = \partial_A\phi\partial_B\phi - g_{AB}\left(\frac{1}{2}\partial_C\phi\partial^C\phi + V(\phi)\right)\,. \tag{3.2}$$

We look for a solution which eventually describes a flat brane embedded in an anti-de Sitter bulk. Hence we consider the metric and scalar field ansatze

$$ds^2|_5 = dw^2 + g(w)\eta_{\mu\nu}dx^\mu dx^\nu \quad , \quad \phi = \phi(w)\,. \tag{3.3}$$

It is then a straightforward exercise to find that the field equations (3.1-2) reduce to

$$\mathcal{T}^w_w = \sigma^w_{[2]w} \qquad \text{with} \qquad \mathcal{T}^w_w = \frac{1}{2}\phi'^2 - v \quad \text{and} \quad \sigma^w_{[2]w} = \mathcal{O} \tag{3.4}$$

$$\mathcal{T}^\mu_\nu = \sigma^\mu_{[2]\nu} \qquad \text{with} \qquad \mathcal{T}^\mu_\nu = -\left[\frac{1}{2}\phi'^2 + v\right]\delta^\mu_\nu \quad \text{and} \quad \sigma^\mu_{[2]\nu} = (\mathcal{O} + \mathcal{LB}')\delta^\mu_\nu \tag{3.5}$$

where

$$\mathcal{O} \equiv -3(k^2-1)(\bar\alpha k^2 + \bar\alpha - 2) \tag{3.6}$$

and

$$\mathcal{LB} \equiv k(\bar\alpha k^2 - 3)\,. \tag{3.7}$$

A prime denotes a derivative with respect to $z \equiv w/\mathcal{L}$, $\mathcal{L}$ being defined by (2.4) ; $v \equiv \mathcal{L}^2 V$, $k \equiv -\frac{1}{2}\frac{g'}{g}$ ($K^\mu_\nu = \frac{k}{\mathcal{L}}\delta^\mu_\nu$ is the extrinsic curvature of the surfaces $w = Const.$), and we have introduced the notation

$$\bar\alpha \equiv \frac{4\alpha}{\mathcal{L}^2}\,. \tag{3.8}$$

(In accordance with the general properties of the Lanczos tensor the Klein-Gordon equation for ϕ is included in (3.4-7), and $\sigma^w_{[2]w} = \mathcal{O}$ is zeroth order in k'. We have gathered in $\mathcal{LB}'$ all the k' terms appearing in $\sigma^\mu_{[2]\nu}$.)

The model must describe a "smooth brane" in an asymptotically AdS_5 bulk with characteristic length scale $\mathcal{L}$. The following requirements must therefore be met. First the bulk stress energy tensor $\mathcal{T}^A_B$ must tend quickly to zero and k to ± 1 as $z \to \pm\infty$, so that the metric (3.3) is asymptotically AdS_5 ; second (and this is crucial) k can vary quickly near $z = 0$ but must not blow up or behave in such a way that g and hence the metric become discontinuous in the thin brane limit.

Now, there exists, for $0 \le \bar\alpha \le 1$, a very simple toy solution of the field equations (3.4-7), satisfying all these requirements, given by, A being a constant

$$k = \tanh Az \qquad \left(\Longrightarrow \qquad g = \frac{1}{(2\cosh Az)^{2/A}}\right) \tag{3.9}$$

which yields

$$\mathcal{O} = -\frac{3}{\cosh^4 Az}[2(1-\bar\alpha)\cosh^2 Az + \bar\alpha] \tag{3.10}$$

$$\mathcal{LB}' = -\frac{3A}{\cosh^2 Az}[(1-\bar\alpha)\cosh^2 Az + \bar\alpha] \tag{3.11}$$

$$\mathcal{LB} = \tanh Az(\bar\alpha\tanh^2 Az - 3) \tag{3.12}$$

as well as

$$v = \frac{3}{2\cosh^4 Az}[(A+4)(1-\bar\alpha)\cosh^2 Az + \bar\alpha(A+2)] \tag{3.13}$$

$$\frac{1}{2}\phi'^2 = \frac{3A}{2\cosh^4 Az}[(1-\bar\alpha)\cosh^2 Az + \bar\alpha]\,. \tag{3.14}$$

In the case $\bar{\alpha} = 0$ (Einstein's theory), and in the "critical" case $\bar{\alpha} = 1$ one obtains $v(\phi)$ in closed form as

$$v(\phi) = \frac{3}{2}(A+4)\cos^2\sqrt{\frac{A}{3}}\phi \qquad \text{for} \qquad \bar{\alpha} = 0 \tag{3.15}$$

$$v(\phi) = \frac{(A+2)}{6}(A\phi^2 - 3)^2 \qquad \text{for} \qquad \bar{\alpha} = 1\,. \tag{3.16}$$

Let us now look at the thin shell limit, that is the $A \to \infty$ limit, of this perfectly smooth solution.

First, from (3.9), $g \to e^{-2|z|}$ and hence the metric tends to its bulk AdS_5 form everywhere.

Second, from (3.10) (3.13-14), $\mathcal{O} = \frac{1}{2}\phi'^2 - v$ tends to zero everywhere, but at $z = 0$ where it remains finite. We have therefore from (3.4) that $\sigma^w_{[2]w} \sim \mathcal{T}^w_w \sim 0$ in the thin brane limit. More precisely

$$\lim_{A\to\infty}\int_I \sigma^w_{[2]w}\, dz = \lim_{A\to\infty}\int_I \mathcal{T}^w_w\, dz = 0$$

where I is an interval centered on $z = 0$ which eventually goes to zero.

Third, using the following (equivalent) definitions of the Dirac distribution[1]

$$\delta(z) \approx \lim_{A\to\infty}\frac{A}{2\cosh^2 Az} \approx \lim_{A\to\infty}\frac{3A}{4\cosh^4 Az} \tag{3.17}$$

we have, from (3.11) (3.13-14), $\mathcal{O} + \mathcal{L}\mathcal{B}' = -\left(\frac{1}{2}\phi'^2 + v\right) \sim \mathcal{L}\mathcal{B}' \to 2(\bar{\alpha}-3)\delta(z)$, and therefore, from (3.5)

$$\lim_{A\to\infty}\sigma^\mu_{[2]\nu} \approx 2(\bar{\alpha}-3)\delta^\mu_\nu\,\delta(z) \approx \lim_{A\to\infty}\mathcal{T}^\mu_\nu\,. \tag{3.18}$$

Comparing this equation with (2.7) (and recalling that $\delta(w) = \mathcal{L}\delta(z)$) we see that we are led to identify

$$T^\mu_\nu \equiv \frac{2}{\mathcal{L}}(\bar{\alpha}-3)\delta^\mu_\nu = -\frac{6}{\mathcal{L}}\left(1 - \frac{4\alpha}{3\mathcal{L}^2}\right)\delta^\mu_\nu \tag{3.19}$$

to the brane stress-energy tensor (or, rather, brane tension in that case).

On this simple toy model we hence recover the "C=1/3" result advocated in [4] [6] [9].[2]

Let us conclude this section with a remark which will be useful in section V. From (3.12) one notes, that

$$\lim_{A\to\infty}\mathcal{L}\mathcal{B} = (\bar{\alpha}-3)\mathcal{S}(z) \tag{3.20}$$

where $\mathcal{S}(z)$ is the sign distribution such that $\mathcal{S}' = 2\delta$. Therefore the brane stress-energy tensor (3.19) is also given by

$$T^\mu_\nu = 2B\delta^\mu_\nu \tag{3.21}$$

[1]Here $f(z) \approx g(z)$ means $\int_I f(z)dz = \int_I g(z)dz$.

[2]Alternative definitions for the brane stress-energy tensor can however be put forward. In [10] the brane stress-energy tensor is defined as the bulk stress-energy tensor $\mathcal{T}^A_B$ evaluated at the particular point z_s (s for "screen"), such that $k'(z_s) = 2/\mathcal{L}$. With such a definition one gets $T^\mu_\nu = -\frac{6}{\mathcal{L}}\left(1 - \frac{4\alpha}{\mathcal{L}^2}\right)\delta^\mu_\nu$, that is the "C=1" result.This "screen" hypersurface $z = z_s$ "stores" the information of the total bending of the brane and can be defined for any smooth function $k'(z)$ which tends to a Dirac distribution $\delta(z)$, hence rendering the result general. Moreover since $\lim_{A\to\infty} z_s = 0$ the screen is inside the domain wall. See [10]for details.

where $\mathcal{LB}'\delta^\mu_\nu$, which contains all the k'-terms, is the dominant part of the Lanczos tensor when $A \to \infty$ and where B is the AdS$_5$ value of $\mathcal{B}$ (3.7) evaluated at $w = 0_+$, that is with $k = +1$.

Of course, it remains to show that the result (3.19) is not model dependent, that is, does not depend on the particular choice made in (3.9) for $k(z)$ (or, equivalently, on the particular choice (3.13-16) for $v(\phi)$), it being understood though that the requirements listed above remain satisfied.[3]

IV. Model independence of the thin flat brane tension

Consider an arbitrary confining potential $v(\phi)$ that is such that

$$v[\phi(z)] = v_0 \delta_A(z) \tag{4.1}$$

where $\delta_A(z)$ is any function which tends to a distribution localized at $z = 0$ when the parameter $A \to \infty$.

We look for a solution of the field equations (3.4-7) such that k is everywhere finite when $A \to \infty$ and tends to ± 1 when $z \to \pm\infty$, and such that k', like v, is "confining", i.e. picked on $w = 0$. Hence, for large A : $\mathcal{O} \ll \mathcal{LB}'$.

Therefore equation (3.4-5) yield, for large A,

$$\frac{1}{2}\phi'^2 \sim v \qquad , \qquad \mathcal{LB}' = [k(\bar{\alpha}k^2 - 3)]' \sim -2v\,. \tag{4.2}$$

Since k is everywhere finite, $\mathcal{LB}$ is also everywhere finite and hence $\mathcal{LB}'$ cannot do else than tend to the Dirac distribution. This implies that $\delta_A(z)$ must be such that $\int_{-\infty}^{+\infty} \delta_A(z) = 1$. Integrating we then get

$$\left[k\left(\bar{\alpha}k^2 - 3\right)\right]_{-\infty}^{+\infty} \sim -2\int_{-\infty}^{+\infty} dz\, v(z) = -2v_0\,. \tag{4.3}$$

Now, $k(\pm\infty) = \pm 1$. Hence

$$v_0 = 3 - \bar{\alpha}\,. \tag{4.4}$$

This result just means that the potential must be "fine-tuned" in order not to introduce a extra, spurious, cosmological constant in the model.

Returning to (3.5) we hence have

$$\sigma^\nu_{[2]\mu} \sim \mathcal{LB}'\delta^\mu_\nu \sim 2(\bar{\alpha} - 3)\delta^\mu_\nu \delta_A(z) \to 2(\bar{\alpha} - 3)\delta^\mu_\nu \delta(z) \tag{4.5}$$

and therefore, from the definition (2.7)

$$T^\mu_\nu = \frac{2}{\mathcal{L}}(\bar{\alpha} - 3)\delta^\mu_\nu = -\frac{6}{\mathcal{L}}\left(1 - \frac{4\alpha}{3\mathcal{L}^2}\right)\delta^\mu_\nu\,. \tag{4.6}$$

[3] Indeed, consider for example the other ansatz : $k = \tanh Az\left(1 + \frac{\beta}{\cosh^2 Az}\right)$ which yields the metric $\ln g = -\ln[(2\cosh Az)^{2/A}] + \frac{\beta}{A\cosh^2 Az}$. If β remains finite when $A \to \infty$ then $\ln g \sim -2|z|\ \forall z$ and this ansatz, as can be easily seen, yields the same brane tension as the $\beta = 0$ case treated in the text. If $\beta = \bar{\beta}A^n$ with $n > 0$ then the brane stress energy tensor is no longer given by (3.19). However such ansatze must be disgarded as the metric $\ln g$ is then no longer continuous (for example, for $n = 2$, $\ln g \sim -2|z| + 2\bar{\beta}\delta(z)$).

Hence we see that the expression for the brane stress-energy tensor obtained in the previous section does not depend on the specific form chosen for the confining potential $v(\phi)$.

As for the bulk stress-energy tensor it is, still for large A

$$\mathcal{T}^{\mu}_{\nu} \sim -2v\,\delta^{\mu}_{\nu} \sim -2\delta^{\mu}_{\nu}(3-\bar{\alpha})\delta_A(z) \quad \text{so that} \quad \lim_{A\to\infty} \mathcal{T}^{\mu}_{\nu} \approx -\frac{6}{\mathcal{L}}\left(1-\frac{4\alpha}{3\mathcal{L}^2}\right)\delta^{\mu}_{\nu}\,\delta(w)\,. \tag{4.7}$$

Hence we check that the expression for the bulk stress-energy tensor obtained in the previous section was not model dependent either.

Let us also, for completeness, give the expressions for the bulk metric and scalar field (at leading order in A).

From (4.1-4) we have that k is the (unique) solution which tends to 1 at $z \to +\infty$ of the cubic equation

$$k(\bar{\alpha}k^2-3) \sim -(3-\bar{\alpha})S_A(z) \tag{4.8}$$

where $S'_A(z) = 2\delta_A(z)$ is such that $S_A(+\infty) = 1$. In the limit $A \to \infty$ S_A tends to the sign distribution $\mathcal{S}$ and

$$\lim_{A\to\infty} k = \mathcal{S} \tag{4.9}$$

and therefore the metric reduces to (2.5).

Finally, from (4.1-2) (4.4) we have that

$$\phi(z) \sim \sqrt{2(3-\bar{\alpha})}\int \sqrt{\delta_A(z)}dz \tag{4.10}$$

and, hence, $v(\phi) = (3-\bar{\alpha})\delta_A(z)$ is known as a function of ϕ, at least implicitely. It is clear that to different functions $\delta_A(z)$ (two examples being displayed in eq (3.15)) will correspond different $v(\phi)$ (e.g. (3.13) or (3.14)). However, whatever the value of $\bar{\alpha}$, these potentials yield the thin brane limit (4.6), (4.7) when $A \to \infty$ (this "loss of information" being the reason why the brane stress-energy tensor becomes model independent in the thin brane limit).

V. Generalization to curved branes

In a Gaussian normal coordinate system adapted to some timelike foliation

$$ds^2|_5 = dw^2 + \gamma_{\mu\nu}(w, x^\rho)dx^\mu dx^\nu \tag{5.1}$$

where

$$\mathcal{K}_{\mu\nu} = -\frac{1}{2}\frac{\partial\gamma_{\mu\nu}}{\partial w} \tag{5.2}$$

is the extrinsic curvature of the surface $w = Const.$, the Einstein Gauss-Bonnet equations

$$\sigma^{A}_{[2]B} = \mathcal{T}^{A}_{B} \tag{5.3}$$

split into

$$\mathcal{T}^{w}_{w} = \sigma^{w}_{[2]w} \qquad \text{with} \qquad \sigma^{w}_{[2]w} = \mathcal{O} \tag{5.4}$$

$$\mathcal{T}^{\mu}_{\nu} = \sigma^{\mu}_{[2]\nu} \qquad \text{with} \qquad \sigma^{\mu}_{[2]\nu} = \mathcal{O}^{\mu}_{\nu} + \frac{\partial\mathcal{B}^{\mu}_{\nu}}{\partial w} \tag{5.5}$$

where $\mathcal{O}$ and $\mathcal{O}^\mu_\nu$ are quartic in the extrinsic curvature and where $\mathcal{B}^\mu_\nu$ is given by [8]

$$\mathcal{B}^\mu_\nu = \mathcal{K}^\mu_\nu - \mathcal{K}\delta^\mu_\nu + 4\alpha\left(\frac{3}{2}\mathcal{J}^\mu_\nu - \frac{1}{2}\mathcal{J}\delta^\mu_\nu - \bar{\mathcal{P}}^\mu{}_{\rho\nu\sigma}\mathcal{K}^{\rho\sigma}\right) \tag{5.6)}$$

with

$$\mathcal{J}^\mu_\nu = -\frac{2}{3}\mathcal{K}^\mu_\rho\mathcal{K}^\rho_\sigma\mathcal{K}^\sigma_\nu + \frac{2}{3}\mathcal{K}\,\mathcal{K}^\mu_\rho\mathcal{K}^\rho_\nu + \frac{1}{3}\mathcal{K}^\mu_\nu(\mathcal{K}.\mathcal{K} - \mathcal{K}^2)\,. \tag{5.7)}$$

and

$$\mathcal{P}_{\mu\rho\nu\sigma} = \mathcal{R}_{\mu\rho\nu\sigma} + (\mathcal{R}_{\mu\sigma}\gamma_{\rho\nu} - \mathcal{R}_{\rho\sigma}\gamma_{\mu\nu} + \mathcal{R}_{\rho\nu}\gamma_{\mu\sigma} - \mathcal{R}_{\mu\nu}\gamma_{\rho\sigma}) - \frac{1}{2}\mathcal{R}\,(\gamma_{\mu\sigma}\gamma_{\rho\nu} - \gamma_{\mu\nu}\gamma_{\rho\sigma}) \tag{5.8}$$

where $\mathcal{R}_{\mu\nu\rho\sigma}$, $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are the Riemann tensor, Ricci tensor and scalar curvature of the surface $w = Const.$. (The fact that all the terms containing a w-derivative of the extrinsic curvature can be gathered in a $w-$ derivative of a tensor $\mathcal{B}^\mu_\nu$ is not trivial and is particular to the Lanczos tensor.)

If all matter is to be confined on the surface $w = 0$ and the metric remain continous then $\mathcal{O}$, $\mathcal{O}^\mu_\nu$ and $\mathcal{B}^\mu_\nu$ will remain finite, while $\frac{\partial\mathcal{B}^\mu_\nu}{\partial w}$ will tend to a delta distribution localized at $w = 0$. More precisely, if the bulk is imposed to be almost anti-de Sitter for all w larger than, say, $w_0 > 0$, with $w_0 \to 0_+$, that is if

$$\mathcal{K}^\mu_\nu \sim \frac{1}{\mathcal{L}}\eta_{\mu\nu} \qquad \forall w > w_0, \qquad w_0 \to 0_+ \tag{5.9}$$

(with $\mathcal{L}$ given by (2.4)), then $\mathcal{O}_w \sim \mathcal{O}^\mu_\nu \sim 0$ and

$$\mathcal{B}^\mu_\nu \sim B^\mu_\nu \mathcal{S}(w) \tag{5.10}$$

with $B^\mu_\nu = \mathcal{B}^\mu_\nu(0+)$ and $\mathcal{S}$ the sign distribution. Consequently

$$\sigma^\mu_{[2]\nu} \sim \frac{\partial\mathcal{B}^\mu_\nu}{\partial w} \to 2B^\mu_\nu\delta(w)\,. \tag{5.11}$$

Hence, from the definition (2.7)

$$T^\mu_\nu = 2B^\mu_\nu \tag{5.12}$$

which generalizes (3.21), is to be identified with the brane matter plus tension stress energy tensor. Therefore, in the general case as well, the brane equations (2.17) obtained by Davis and Gravanis-Willison are recovered.

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